Solution 1

1. Let f be a 2π-periodic function which is integrable over $[-\pi,\pi]$. Show that it is integrable over any finite interval and
$$\int_I f(x)dx = \int_J f(x)dx,$$
where I and J are intervals of length 2π.

Solution It is clear that f is also integrable on $[n\pi,(n+2)\pi]$, $n\in\mathbb{Z}$, so it is integrable on the finite union of such intervals. As every finite interval can be a subinterval of intervals of this type, f is integrable on any $[a,b]$. To show the integral identity it suffices to take $J=[-\pi,\pi]$ and $I=[a,a+2\pi]$ for some real number a. Since the length of I is 2π, there exists some n such that $n\pi\in I$ but $(n+2)\pi$ does not belong to the interior of I. We have
$$\int_a^{a+2\pi} f(x)dx = \int_a^{n\pi} f(x)dx + \int_{n\pi}^{a+2\pi} f(x)dx.$$
Using
$$\int_a^{n\pi} f(x)dx = \int_{a+2\pi}^{(n+2)\pi} f(x)dx$$
(by a change of variables), we get
$$\int_a^{a+2\pi} f(x)dx = \int_{a+2\pi}^{(n+2)\pi} f(x)dx + \int_{n\pi}^{a+2\pi} f(x)dx = \int_{n\pi}^{(n+2)\pi}.$$
Now, using a change of variables again we get
$$\int_{n\pi}^{(n+2)\pi} f(x)dx = \int_{-\pi}^{\pi} f(x)dx.$$

2. Verify that the Fourier series of every even function is a cosine series and the Fourier series of every odd function is a sine series.

Solution Write
$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx).$$
Suppose $f(x)$ is an even function. Then, for $n\geq 1$, we have
$$\pi b_n = \int_{-\pi}^{\pi}\sin nx f(x)dx = \int_{-\pi}^{0}\sin nx f(x)dx + \int_0^{\pi}\sin nx f(x)dx\ .$$
By a change of variable and using $f(-x)=f(x)$ since $f(x)$ is an even function,
$$\int_{-\pi}^{0}\sin nx f(x)dx = \int_0^{\pi}\sin(-nx)f(-x)dx = -\int_0^{\pi}\sin nx f(x)dx,$$
one has
$$\pi b_n = -\int_0^{\pi}\sin nx f(x)dx + \int_0^{\pi}\sin nx f(x)\,dx = 0.$$
Hence the Fourier series of every even function f is a cosine series.

Now suppose $f(x)$ is an odd function. Then, for $n\geq 1$, we have
$$\pi a_n = \int_{-\pi}^{\pi}\cos nx f(x)dx = \int_{-\pi}^{0}\cos nx f(x)dx + \int_0^{\pi}\cos nx f(x)dx\ .$$

By a change of variable and using $f(-x) = -f(x)$ since $f(x)$ is an odd function,

$$\int_{-\pi}^{0} \cos nx f(x)dx = \int_{0}^{\pi} \cos(-nx)f(-x)dx = -\int_{0}^{\pi} \cos nx f(x)dx,$$

one has

$$\pi a_n = -\int_{0}^{\pi} \cos nx f(x)dx + \int_{0}^{\pi} \cos nx f(x)dx = 0 \ , \quad \forall n \geq 0 \ .$$

3. Here all functions are defined on $[-\pi, \pi]$. Verify their Fourier expansion and determine their convergence and uniform convergence (if possible).

   (a)
   $$x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx,$$

   (b)
   $$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x,$$

   (c)
   $$f(x) = \begin{cases} 1, & x \in [0, \pi] \\ -1, & x \in [-\pi, 0] \end{cases} \sim \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x,$$

   (d)
   $$g(x) = \begin{cases} x(\pi - x), & x \in [0, \pi) \\ x(\pi + x), & x \in (-\pi, 0) \end{cases} \sim \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x.$$

Solution

(a) Consider the function $f_1(x) = x^2$. As $f_1(x)$ is even, its Fourier series is a cosine series and hence $b_n = 0$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi}\frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{\pi^2}{3},$$

and by integration by parts,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx \\ &= \frac{1}{n\pi}x^2 \sin nx\Big|_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x \sin nx dx \\ &= \frac{2}{n^2\pi}x \cos nx\Big|_{-\pi}^{\pi} - \frac{2}{n^2\pi}\int_{-\pi}^{\pi} \cos nx dx \\ &= 4\frac{(-1)^n}{n^2}. \end{aligned}$$

For $n \geq 1$,

$$|a_n| = |-4\frac{(-1)^{n+1}}{n^2}| \leq \frac{4}{n^2}.$$

We conclude that the Fourier series converges uniformly by the Weierstrass M-test.

(b) Consider the function $f_2(x) = |x|$. As $f_2(x)$ is even, its Fourier series is a cosine series and hence $b_n = 0$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| dx = \frac{1}{2\pi}\frac{x^2}{2}\bigg|_{-\pi}^{\pi} = \frac{\pi}{2},$$

and by integration by parts,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx \\ &= \frac{2}{n\pi} x\sin nx\bigg|_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}\sin nx dx \\ &= -\frac{2}{n^2\pi}\cos nx\bigg|_0^{\pi} \\ &= -2\frac{[(-1)^n - 1]}{n^2\pi}. \end{aligned}$$

For $n \geq 1$,

$$|a_n| = |2\frac{[(-1)^n - 1]}{n^2\pi}| \leq \frac{4}{\pi n^2}.$$

We conclude that the Fourier series converges uniformly by the Weierstrass M-test.

(c) As $f(x)$ is odd, its Fourier series is a sine series and hence $a_n = 0$.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi}\sin nx dx \\ &= \frac{2}{n\pi}\cos nx\bigg|_0^{\pi} \\ &= 2\frac{[(-1)^n - 1]}{n\pi}. \end{aligned}$$

Now we consider the convergence of the series $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$. Fix $x \in (-\pi, 0) \cup (0, \pi)$, Using the elementary formula

$$\sum_{n=1}^{N}\sin(2n-1)x = \frac{\sin^2(N+1)x}{\sin x},$$

one has that the partial sums $|\sum_{n=1}^{N}\sin(2n-1)x| = |\frac{\sin^2(N+1)x}{\sin x}| \leq |\frac{1}{\sin x}|$ are uniformly bounded. This also holds for $x = 0$, in which case $|\sum_{n=1}^{N}\sin(2n-1)0| = 0$. Furthermore, the coefficients $1/(2n-1)$ decreases to 0. We conclude that the Fourier series converges pointwisely by Dirichlet's test.

(d) As $g(x)$ is odd, its Fourier series is a sine series and hence $a_n = 0$. By integration by parts,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx dx \\ &= -\frac{2}{n\pi}x(\pi - x)\cos nx\bigg|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}(\pi - 2x)\cos nx dx \\ &= \frac{2}{n^2\pi}(\pi - 2x)\sin nx\bigg|_0^{\pi} + \frac{4}{n^2\pi}\int_0^{\pi}\sin nx dx \\ &= -\frac{4}{n^3\pi}\cos nx\bigg|_0^{\pi} \\ &= -\frac{4}{n^3\pi}[(-1)^n - 1]. \end{aligned}$$

As

$$|b_n| \le \frac{8}{\pi n^3},$$

we conclude that the Fourier series converges uniformly by the Weierstrass M-test.

4. Let f be a 2π-periodic function whose derivative exists and is integrable on $[-\pi, \pi]$. Show that its Fourier series decay to 0 as $n \to \infty$ without appealing to Riemann-Lebesgue Lemma. Hint: Use integration by parts to relate the Fourier coefficients of f to those of f'.

   Solution Performing integration by parts yields

$$\pi a_n = \int_{-\pi}^{\pi} f(x) \cos nx dx = -\frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin nx dx \ .$$

   Therefore,

$$\pi |a_n| \le \frac{1}{n} \int_{-\pi}^{\pi} |f'(x)| dx \to 0 \ , \quad n \to \infty \ .$$

   Similarly the same result holds for b_n.

5. Let f be a continuous 2π-periodic function. Show that its Fourier series decay to 0 as $n \to \infty$ without appealing to Riemann-Lebsegue lemma. Hint: Establish the formula

$$2\pi a_n = \int_{-\pi}^{\pi} [f(y) - f(y + \pi/n)] \cos ny \, dy \ ,$$

   using Problem 1.

   Solution Setting $y = z + \pi/n$, we have

$$\begin{aligned} \pi a_n &= \int_{-\pi}^{\pi} f(y) \cos ny \, dy \\ &= \int_{-\pi+\pi/n}^{\pi+\pi/n} f(z + \pi/n) \cos(z + \pi/n) \, dz \\ &= \int_{-\pi+\pi/n}^{\pi+\pi/n} f(z + \pi/n)(-\cos z) \, dz \\ &= -\int_{-\pi}^{\pi} f(y + \pi/n) \, dy \ . \end{aligned}$$

   It follows that

$$2\pi a_n = \int_{-\pi}^{\pi} [f(y) - f(y + \pi/n)] \cos ny \, dy \ .$$

   By assumption f is continuous on $[-\pi, \pi]$, it is uniformly continuous there. For $\varepsilon > 0$, there is some n_0 such that $|f(y) - f(y + \pi/n)| < \varepsilon$ for all y and $n \ge n_0$. It follows that $|a_n| < \varepsilon$ for all $n \ge n_0$, that is, $a_n \to 0$ as $n \to \infty$. Similarly we can show $b_n \to 0$.

6. Let g be an integrable T-periodic function. Show that for any integrable function f on $[a, b]$,

$$\lim_{n \to \infty} \int_a^b f(x) g(nx) \, dx = \frac{1}{T} \int_0^T g(x) \, dx \int_a^b f(x) \, dx \ .$$

   Suggestion: Consider the special case $\int_0^T g(x) dx = 0$ first.

Solution First assume $\int_0^T g(x)dx = 0$. Divide an interval $I = [c, d]$ into the union of $[c + (k-1)T/n, c + kT/n], k = 1, \cdots, N$ so that $0 < d - (c + NT/n) < T/n$. Then

$$\int_c^d g(nx)\,dx = \sum_k^N \int_{c+(k-1)T/n}^{c+kT/n} g(nx)\,dx + \int_{c+NT/n}^d g(nx)\,dx \ .$$

Since

$$\int_{c+(k-1)T/n}^{c+kT/n} g(nx) = \int_{nc+(k-1)T}^{nc+kT} g(y)\,dy = \int_0^T g(y)\,dy = 0 \ ,$$

for each k,

$$\Big|\int_c^d g(nx)\,dx\Big| = \Big|\int_{c+NT/n}^d g(nx)\,dx\Big| \le \int_{c+NT/n}^{c+(N+1)T/n} |g(nx)|\,dy = \frac{1}{n}\int_0^T |g(y)|\,dy,$$

which clearly tends to 0 as $n \to \infty$.

From the form of a step function we see that $\int_a^b s(x)g(nx)\,dx \to 0$ as $n \to \infty$. By approximating f by step functions from below as in the proof of the R-L lemma, we see that

$$\int_a^b f(x)g(nx)\,dx \to 0,$$

as $n \to \infty$ for every integrable function f.

Now, for any integrable T-periodic function g, the function $h(x) = g(x) - \frac{1}{T}\int_0^T g(y)\,dy$ satisfies $\int_a^b h(x)\,dx = 0$. From

$$\lim_{n\to\infty} \int_a^b f(x)h(nx)\,dx = 0$$

we draw the desired conclusion.

Remark This problem extends Riemann-Lebsegue Lemma without much additional effort.

7. A sequence of integrable functions $\{g_n\}_1^\infty$ on $[a, b]$ is called an orthonormal family if (a) $\int_a^b g_n(x)g_m(x)\,dx = 0$ for $n \neq m$ and $\int_a^b g_n^2(x)\,dx = 1$ for all n. Show that whenever $f(x) = \sum_{n=1}^\infty c_n g_n(x)$ holds, $c_n = \int_a^b f(x)g_n(x)\,dx$ for all n. Is the family $\{1, \cos nx, \sin nx\}$ orthonormal on $[-\pi, \pi]$?

   Solution If $f(x) = \sum_{n=1}^\infty c_n g_n$, multiply it by g_m and (formally) integrate to get

   $$\int_a^b f(x)g_m(x)\,dx = \sum_{n=1}^\infty \int_a^b c_n g_m(x)g_n(x)\,dx = c_m \ , \forall m \ .$$

   The family $\{1, \cos x, \sin x, \cdots,\}$ satisfies condition (a) but not (b). Indeed the normalized one

   $$\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \cdots,\}$$

   forms an orthonormal family.